Elementary Mathematical Logic and Related Topics

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CHAPTER 1: LOGIC

"If it was so, it might be; and if it were so, it would be; but as it isn't, it ain't. That's logic."
- Lewis Carroll, Alice in Wonderland

I. STATEMENTS AND STANDARD FORMS:

A. DEFINITION OF A STATEMENT

Definition 1 A ***statement*** is any declarative sentence that is either true (T) or false (F), but not both. We refer to T or F as the ***truth value*** of the statement. Statements are usually denoted by lower case letters (for example: *p, q, r*, …)

Example 1 Let p be the sentence "Mexico is not a state in the USA." Let q be the sentence "The world is flat." These are both statements because we can assign to them a truth value. In this case, p is true and q is false.

The sentence "Stand up" or "Follow the leader" cannot be used as statements. These are commands instead of a declarative sentence and therefore cannot be classified as true or false.

B. NEGATION

Definition 2 The ***negation*** of a statement p, denoted by ~p (read as "not p") is the statement whose truth value is the opposite of the truth value of p.

Example 2 Using statement p of Example 1, we write ~p as "It is not the case that Mexico is not a state in the USA." or more simply, "Mexico is a state in the USA." Similarly, $\sim q$ is written as "It is not the case that the world is flat." or "The world is not flat." Notice that these negations are also statements that can be given truth values. In these examples, $\sim p$ is false and $\sim q$ is true.

TRUTH TABLES

A truth table can be set up to show the truth values for different statements. Usually, each column represents a statement, and the rows below show the different truth values possible. For example, since we know that if p is true, then $\sim p$ is false, we can set up the following truth table to show that relationship:

Truth table for $\sim p$

p	$\sim p$
T	
F	

Remark: The negations of statements will be referred to as being in ***standard form***. Although the negations of statements may be written in several ways, there is usually one form that is preferred when you are working with logic problems. We will learn these standard forms as different types of statements are introduced.

Example 3 The standard form of the negation of the statement "Mexico is not a state in the USA." is "Mexico is a state in the USA."

Example 4 The standard form of the negation of the statement "Texas is not a state." is "Texas is a state."

Exercise 1: Write the negation in standard form for each of the following statements.

Statement: *a*: Dallas is the capitol of Oklahoma.

Negation: *~a:* ____

Statement: *b*: An orange is a fruit.

Negation: *~b:* ____

Statement: *c*: An apple is not blue.

Negation: *~c:* ____

Give the truth values (True or False) of each of the above statements.

a ____________ *b* _____________ *c* ______________

~a ___________ *~b* ____________ *~c* _____________

C. CONJUNCTION

Definition 3 The ***conjunction*** of p and q, $p \wedge q$ (read as "p and q"), is the statement that is true if and only if both p and q are true. In other words, if someone told you that he owned a dog and a cat, you would only consider that statement true if he really did own both animals, not just one or the other.

Example 5 Let p be the statement "Santa is jolly." Let q be the statement "Christmas is on July 4th." Let r be the statement "George Washington was the first U.S. president." The conjuction of p and q is the statement "Santa is jolly and Christmas is on July 4th." $p \wedge r$ is written as "Santa is jolly and George Washington was the first U.S. president." Likewise, $q \wedge r$ is written as "Christmas is on July 4th and George Washington was the first U.S. president."

DETERMINING THE TRUTH VALUE OF A CONJUCTION When given statements p and q, the truth value for $p \wedge q$ is true whenever each of the component statements is true. In any other case, the truth value is false.

Truth table for $p \wedge q$

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

In example 5, $p \wedge q$ is___________, $p \wedge r$ is _________, and $q \wedge r$ is ___________

D. DISJUNCTION

Definition 4. The ***disjunction*** of p and q, $p \vee q$ (read as "p or q"), is the statement that is true if and only if at least one of p and q is true.

Example 6. Let p, q, and r be the same statements as in Example 5. The disjunction of p and q is the statement "Santa is jolly or Christmas is on July 4th." If either of those statements are true, then the disjunction is true. $p \vee r$ is written as "Santa is jolly or George Washington was the first U.S. president." $q \vee r$ is written as "Christmas is on July 4th or George Washington was the first U.S. president."

DETERMINING THE TRUTH VALUE OF A DISJUNCTION Given statements p and q. The truth value for $p \vee q$ is true whenever either of the component statements is true; otherwise, the truth value is false.

Truth table for $p \vee q$

p	q	$p \vee q$
T	T	T
T	F	T

F F	T F	T F

In example 6, $p \vee q$ is __________, $p \vee r$ is ____________, and $q \vee r$ is _____________.

However, the disjunction $\sim p \vee q$ is false.

Exercise 2 Given the statements

a: Abraham Lincoln was the 16th U.S. president.
b: A lemon is tart.
c: Valentine's Day is in December.

i. Write the conjunction of a and b.

$a \wedge b$: ____

ii. Write the disjunction of a and c.

$a \vee c$: ____

iii. Write the conjunction of $\sim a$ and b.

$\sim a \wedge b$: ____

iv. Write the disjunction of $\sim c$ and $\sim a$

$\sim c \vee \sim a$: ____

Write the truth value of each statement given above.

i. ____________ *ii.* ______________ *iii.* _____________ *iv.* ______________

E. USE OF PARENTHESES

Parentheses are used to group statements together. For example, the two statements $(p \vee q) \wedge r$ and $p \vee (q \wedge r)$ may appear to be the same to the casual observer. However, their truth tables are different:

p	q	r	$(p \vee q) \wedge r$	$p \vee (q \wedge r)$
T	T	T		
T	T	F		
T	F	T		
T	F	F		
F	T	T		
F	T	F		
F	F	T		
F	F	F		

This demonstrates that $(p \vee q) \wedge r$ is not the same thing as $p \vee (q \wedge r)$.

Remember that the operation inside the parentheses must be done *first*.

Sometimes, there will be a set of parentheses inside of another set of parentheses; or, a set of parentheses inside of a set of brackets. This means that you should do the operation on the *innermost* set of parentheses first.

Example 7 Let p, q, and r be statements with the following truth values: p is true, q is true, and r is false. Find the truth value of the following statement:

$$\left[p \vee (q \wedge r)\right] \vee \sim q$$

Step 1: $(q \wedge r)$ will have a truth value of false.

Step 2: Since p is true and $(q \wedge r)$ is false, $\left[p \vee (q \wedge r)\right]$ will have a truth value of true.

Step 3: Since $\left[p \vee (q \wedge r)\right]$ is true and $\sim q$ is false, $\left[p \vee (q \wedge r)\right] \vee \sim q$ will be true.

Exercise 3 Construct truth tables for $p \wedge (q \vee r)$ and $(p \wedge q) \vee r$.

So these statements have the same truth tables? _______ Are they equivalent? __________

F. CONDITIONAL STATEMENT

Definition 5 The ***conditional statement*** , $p \to q$ (read as "If p, then q") is the statement that is true unless p is true and q is false. p is called the ***antecedent*** and q is called the ***consequent***. If $p \to q$ is true, then sometimes p is called the ***hypothesis*** and q is called the ***conclusion***.

Example 8 Let p, q, and r be the same statements as in Example 5. $p \to q$ is written as "If Santa is jolly, then Christmas is on July 4th." $q \to r$ is written as "If Christmas is on July 4th, then George Washington was the first U.S. President." $p \to r$ is written as "If Santa is jolly, then George Washington was the first U.S. president."

Negations can also be used in conditional statements. $\sim p \to r$ is written as "If Santa is not jolly, then George Washington was the first U.S. president." $\sim r \to q$ is written as "If George Washington was not the first U.S. president, then Christmas is on July 4th." $p \to \sim q$ is written as "If Santa is jolly, then Christmas is not on July 4th."

DETERMINING THE TRUTH VALUE OF A CONDITIONAL STATEMENT

Given statements p and q. The truth value for $p \to q$ is true unless p is true and q is false. If p is true and q is false, then $p \to q$ is false.

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

In example 8, $p \to q$ is ___________, $q \to r$ is ___________, $p \to r$ is __________, and $\sim p \to r$ is __________.

VARIATIONS OF THE CONDITIONAL STATEMENT $p \to q$

Definition 6 The ***contrapositive*** of $p \to q$ is $\sim q \to \sim p$.

Definition 7 The ***converse*** of $p \to q$ is $q \to p$.

Definition 8 The ***inverse*** of $p \to q$ is $\sim p \to \sim q$.

Example 9 Consider the conditional statement: "If Santa is jolly, then Christmas is on July 4th." The ***contrapositive*** is "If Christmas is not on July 4th, then Santa is not jolly." The ***converse*** is "If Christmas is on July 4th, then Santa is jolly." The ***inverse*** is "If Santa is not jolly, then Christmas is not on July 4th."

TRUTH TABLE FOR CONDITIONAL STATEMENTS:

p	q	$\sim p$	$\sim q$	$p \to q$	$\sim q \to \sim p$	$q \to p$	$\sim p \to \sim q$
T	T	F	F	T	T	T	T
T	F	F	T	F	F	T	T
F	T	T	F	T	T	F	F
F	F	T	T	T	T	T	T

QUESTION: Given the conditional statement $p \to q$ and considering its contrapositive, converse, and inverse, which of these statements have the same (identical) truth table?

____

ALTERNATE WAYS OF EXPRESSING A CONDITIONAL STATEMENT

Given the conditional statement $p \to q$, the following expressions mean the same thing:

"If *p*, then *q*."

"*p* is a sufficient condition for *q*."

"*p* only if *q*."

"*q* is a necessary condition for *p*."

"*q* if *p*."

Example 10 Let p be the statement "A triangle is isosceles." Let q be the statement "The triangle has at least two congruent sides." $p \to q$ is written "If a triangle is isosceles, then the triangle has at least two congruent sides." However $p \to q$ may also be written:

A triangle is isosceles is a sufficient condition that the triangle has at least two congruent sides.

A triangle is isosceles only if the triangle has at least two congruent sides.

A triangle has at least two congruent sides is a necessary condition that the triangle is isosceles.

A triangle has at least two congruent sides if the triangle is isosceles.

G. BICONDITIONAL STATEMENT

Definition 9 The ***biconditional statement*** $p \leftrightarrow q$ means $(p \to q) \wedge (q \to p)$, or "if *p* then *q*, and if *q* then *p*." We read $p \leftrightarrow q$ as "***p* if and only if *q***."

Example 11 Let *p* and *q* be the statements of Example 10. $p \leftrightarrow q$ is written "A triangle is isosceles if and only if the triangle has at least two congruent sides." $p \leftrightarrow q$ may also be written "If a triangle is isosceles, then the triangle has at least two congruent sides; and if a triangle has at least two congruent sides, then the triangle is isosceles."

DETERMINING THE TRUTH VALUE OF A BICONDITIONAL STATEMENT

Given statements p and q. The truth values for $p \leftrightarrow q$ are in the table below.

Truth Table for $p \leftrightarrow q$

p	q	$p \rightarrow q$	$q \rightarrow p$	$p \leftrightarrow q$
T	T	T	T	T
T	F	F	T	F
F	T	T	F	F
F	F	T	T	T

Note: $p \leftrightarrow q$ will be true whenever the statements p and q are both true or both false; otherwise, $p \leftrightarrow q$ will be false.

OTHER WAYS OF EXPRESSING THIS BICONDITIONAL STATEMENT ARE THE FOLLOWING:

A triangle is isosceles is a sufficient condition that the triangle has at least two congruent sides; and a triangle has at least two congruent sides is a sufficient condition that the triangle is isosceles.

A triangle has at least two congruent sides is a necessary condition that the triangle is isosceles; and a triangle has at least two congruent sides is a sufficient condition that the triangle is isosceles.

A triangle has at least two congruent sides is a necessary and sufficient condition that the triangle is isosceles.

Theorem $p \rightarrow q$ is true if and only if $p \vee q$ has the same truth table as q.

Note: A **theorem** is a statement of mathematical truth that can be proven.

Proof of Theorem:

First part of proof: It must be shown that if $p \rightarrow q$ is true, then $p \vee q$ and q have the same truth table.

Second part of proof: It must also be shown that if $p \vee q$ and q have the same truth table, then $p \rightarrow q$ is a true statement.

First part of proof:

p	q	$p \vee q$	$p \rightarrow q$
T	T		
T	F		
F	T		
F	F		

Second part of proof:

p	q	$p \vee q$	$p \rightarrow q$
T	T		
T	F		
F	T		
F	F		

Exercise 4 Given the statemtents:

p: The moon is full. q: The tide is high.

i. Write the conditional statement with *p* as the antecedent and *q* as the consequent.

ii. Write the converse of *i.*

iii. Write the inverse of *i.*

iv. Write the contrapositive of *i*.

v. Write $p \leftrightarrow q$ in two different ways.

a.

b.

NOTE: In the original definition of *statement*, it was stated that the only sentence that can be used as a statement in logic is a sentence that can be assigned a value of true or false. However, in these examples and exercises, there will be the occasional use of a sentence that cannot be absolutely categorized as true or false without more information. For example, in Exercise 4, "*The moon is full*." is used as a statement. It is impossible to know if that sentence is true or false as written. However, it can be used as a statement since, on a specific night, it could be definitively assigned a value of true or false. Therefore, it is acceptable to use as statements those sentences that could, given a specific example, be assigned a truth value.

II. LOGICAL EQUIVALENCE

Definition 10 Two statements are said to be ***logically equivalent*** if and only if they have the same (identical) truth table.

Notation: $p \equiv q$ (Read "*p* is logically equivalent to *q*.")

$p \not\equiv q$ (Read "*p* is not logically equivalent to *q*.")

Remark: $p \equiv q$ is a relation between statements *p* and *q*. This relation is called a ***logical equivalence*** and is not a new statement compounded from statements *p* and *q*.

Example 12 Given statements *p* and *q*:

1. $p \to q$ and $\sim q \to \sim p$ are logically equivalent; that is, $p \to q \equiv \sim q \to \sim p$.
2. $q \to p \equiv \sim p \to \sim q$ **Contrapositive**
3. $\sim(\sim p) = p$ **Double Negation**
4. $p \to q \equiv \sim p \vee q$ **Conditional to Disjunction**
5. $p \wedge q \equiv q \wedge p, \quad p \vee q \equiv q \vee p$ **Commutative Property**
6. $p \to q \not\equiv q \to p$
7. $\sim(p \wedge q) \equiv \sim p \vee \sim q$ **DeMorgan's Law**
8. $\sim(p \vee q) \equiv \sim p \wedge \sim q$ **DeMorgan's Law**
9. $\sim(p \to q) \equiv p \wedge \sim q$ **Negation of a Conditional Statement**

PROPERTIES OF LOGICAL EQUIVALENCE: Let *p*, *q*, and *r* be any three statements.

Reflexive Property $p \equiv p$

Symmetric Property If $p \equiv q$, then $q \equiv p$

Transitive Property If $p \equiv q$ and $q \equiv r$, then $p \equiv r$

Axiom of Substitution Given any statement, part of the statement can be substituted (replaced) by a logically equivalent statement without effecting the truth value of the original statement.

Note: The first three properties of logical equivalence follow from Definition 11.

Exercise 5

i. Complete the truth table

p	q	r	$q \wedge r$	$p \wedge q$	$q \vee r$	$p \wedge r$	$p \wedge (q \vee r)$	$p \vee (q \wedge p)$	$(p \wedge q) \vee (p \wedge r)$
T	T	T							
T	T	F							
T	F	T							
T	F	F							
F	T	T							
F	T	F							
F	F	T							
F	F	F							

List the statements from the truth table that are logically equivalent.

ii. Use truth tables to prove: $\sim(p \wedge q) \equiv \sim p \vee \sim q$ (DeMorgan's Law)

III. STANDARD FORM OF COMPOUND STATEMENTS

In standard form, the ***negation of a conjunction*** is written as the disjunction of the negations of the two component statements (See Example 12.7)

$$\sim(p \wedge q) \equiv \sim p \vee \sim q$$

Also, $\sim p$ and $\sim q$ must be written in standard form. For example, to write the statement "It is not the case that the cheetah is a dog and the cheetah is fast." in standard form, you would take the negation of each statement in standard form, and join them as a disjunction. The statement would then become "The cheetah is not a dog or the cheetah is not fast."

Likewise, the ***negation of a disjunction*** is written in standard form as the *conjunction* of the negations of the two component statements (See Example 12.8)

$$\sim(p \vee q) \equiv \sim p \wedge \sim q$$

For example, "The sun is not a planet and Jupiter is not a star." is the standard form of the statement "It is not the case that the sun is a planet or Jupiter is a star."

Example 13 The following statement (1) is changed to an equivalent statement (3) in standard form:

1. It is not the case that the polar bear is white and the soccer ball is not round.

2. It is not the case that the polar bear is white or it is not the case that the soccer ball is not round.

3. The polar bear is not white or the soccer ball is round.

Statement (3) is in standard form. Statement (2) was obtained by applying DeMorgan's Law to statement (1).

The ***"double negation " of a statement*** written in standard form is just the statement with no negations (Example 12.3)

$$\sim(\sim p) = p$$

For example, *"Jonas Salk invented the polio vaccine" is the standard form of "It is not true that Jonas Salk did not invent the polio vaccine."*

The ***negation of a conditional statement*** is written in standard form as the conjunction of the antecedent and the negation of the consequent (Example 12.9)

$$\sim(p \rightarrow q) \equiv p \wedge \sim q$$

For example, the standard form of the statement "It is not the case that if Tiger Woods is a golfer, then Michael Jordan is a golfer" is "Tiger Woods is a golfer and Michael Jordan is not a golfer."

Example 14 The following statement (1) is changed to an equivalent statement (3) in standard form:

1. It is not the case that if Quebec is not in Mexico, then Houston is in Canada.

2. Quebec is not in Mexico, and it is not the case that Houston is in Canada.

3. Quebec is not in Mexico and Houston is not in Canada.

Statement (3) is in standard form. The intermediate statement (2) follows from the application of Example 12.9 to statement (1).

Exercise 6 Given the statements

p: Abraham Lincoln was the 16^{th} U.S. President

q: A lemon is tart.

r: Valentine's Day is in December.

a. Write in standard form the negation of the conjunction of p and q.

b. Write in standard form the negation of the disjunction of p and q.

c. Write in standard form the negation of the disjunction of q and r.

d. Write in standard form the negation of the conjunction of p and r.

e. Write in standard form the negation of $p \rightarrow q$.

IV. USING TRUTH TABLES

Sometimes, it may seem that it would be difficult to develop a truth table for a very complex statement with multiple operators and different kinds of bracketing. However, if you will approach the problem systematically, numbering each step as you go, you should find the process much easier.

First, write out the statement you wish to analyze. The statement will consist of basic statements, or propositions, that are denoted by lower-case letters. These propositions will be joined by different operators in a particular order, as determined by the parentheses and brackets. Allow each proposition and each operator to have its own column.

Then, determine the truth values for each column, **numbering** each step at the bottom of the column as completed. Determine the truth values in the following way:

First, establish the possible combinations of truth values for the basic propositions. For example, if you have statement *p* and statement *q*, there would be four different combinations of truth values to list: TT, TF, FT, FF. For *n* different statements, there will be 2^n different combinations of truth values. Each row will represent a different set of possible truth value combinations. Therefore, a proposition should have the same truth value throughout a row, no matter how many times it is listed.

Next, check to see if there are negations of the basic propositions, and fill in those truth values. Be sure that these are negations of only the propositions – they should not be separated from the propositions by any operators or bracketing.

Finally, fill in the truth values of the compound statements formed by the different operators. Be sure to work from the most interior parentheses outwards, numbering each step as you complete the column. Write the truth value under the operator that defines the compound statement. The truth values for the entire statement will be found in the column with the largest step number.

Example 15 Suppose we want to find the truth values for the following statement:

$$\left[(p \to q) \wedge (q \to r)\right] \vee (p \to r)$$

To set up the appropriate truth table, write the needed statement at the top of the table, allowing a column for each given proposition and each operator:

$$\left[(p \quad \to \quad q) \quad \wedge \quad (q \quad \to \quad r)\right] \quad \vee \quad (p \quad \to \quad r)$$

T	**T**	**T**	**T**	**T**	**T**
T	**T**	**T**	**F**	**T**	**F**
T	**F**	**F**	**T**	**T**	**T**
T	**F**	**F**	**F**	**T**	**F**
F	**T**	**T**	**T**	**F**	**T**
F	**T**	**T**	**F**	**F**	**F**
F	**F**	**F**	**T**	**F**	**T**
F	**F**	**F**	**F**	**F**	**F**
1	**1**	**1**	**1**	**1**	**1**

<u>Step 1</u>: List the possible truth values for each given proposition. In this case, there are three given propositions, so there will be 2 x 2 x 2 = 8 different combinations of truth values for these three statements. Be sure that the values are consistent across each row – if *p* is true in row 1, *p* will be true each time it appears in row 1.

$$\left[(p \quad \to \quad q) \quad \wedge \quad (q \quad \to \quad r)\right] \quad \vee \quad (p \quad \to \quad r)$$

T	**T**	T	T	**T**	T	T	**T**	T
T	**T**	T	T	**F**	F	T	**F**	F
T	**F**	F	F	**T**	T	T	**T**	T
T	**F**	F	F	**T**	F	T	**F**	F
F	**T**	T	T	**T**	T	F	**T**	T
F	**T**	T	T	**F**	F	F	**T**	F
F	**T**	F	F	**T**	T	F	**T**	T
F	**T**	F	F	**T**	F	F	**T**	F
1	**2**	1	1	**2**	1	1	**2**	1

<u>Step 2</u>: Find the truth value of the new statements formed by the operators. Since there are no immediate negations of the propositions, we will begin to work with the operators from the most interior parentheses, and then work outward.

$$\left[(p \quad \to \quad q) \quad \wedge \quad (q \quad \to \quad r)\right] \quad \vee \quad (p \quad \to \quad r)$$

T	T	T	**T**	T	T	T	T	T	T
T	T	T	**F**	T	F	F	T	F	F
T	F	F	**F**	F	T	T	T	T	T
T	F	F	**F**	F	T	F	T	F	F
F	T	T	**T**	T	T	T	F	T	T
F	T	T	**F**	T	F	F	F	T	F
F	T	F	**T**	F	T	T	F	T	T
F	T	F	**T**	F	T	F	F	T	F
1	2	1	**3**	1	2	1	1	2	1

<u>Step 3</u>: After the truth values defined by the interior parentheses are determined, we can consider the statement formed within the outer brackets. We must use the truth values that we found in step 2 (columns 2 and 6) to find the values for the new statement.

$$\left[(p \quad \to \quad q) \quad \wedge \quad (q \quad \to \quad r)\right] \quad \vee \quad (p \quad \to \quad r)$$

T	T	T	T	T	T	T	**T**	T	T	T
T	T	T	F	T	F	F	**F**	T	F	F
T	F	F	F	F	T	T	**T**	T	T	T
T	F	F	F	F	T	F	**F**	T	F	F
F	T	T	T	T	T	T	**T**	F	T	T

<u>Step 4</u>: Finally, we can find the truth values of the entire statement using the values found in column 4 and column 10.

F	T	T	F	T	F	F	**T**	F	T	F
F	T	F	T	F	T	T	**T**	F	T	T
F	T	F	T	F	T	F	**T**	F	T	F
1	2	1	3	1	2	1	**4**	1	2	1

Note that $\left[(p \to q) \wedge (q \to r)\right] \vee (p \to r) \equiv (p \to r)$

<u>Example 16</u> Construct a truth table for the following statement: $\sim(p \to q) \wedge \sim p$

$\sim$	$(p$	$\to$	$q)$	$\wedge$	$\sim$	p
	T		**T**			**T**
	T		**F**			**T**
	F		**T**			**F**
	F		**F**			**F**
	1		**1**			**1**

<u>Step 1:</u> List the possible truth values for each proposition.

$\sim$	$(p$	$\to$	$q)$	$\wedge$	$\sim$	p
	T		T		**F**	T
	T		F		**F**	T
	F		T		**T**	F
	F		F		**T**	F
	1		1		**2**	1

<u>Step 2:</u> Fill in the truth values for the negation of proposition *p*. Notice that the first negation, that of the conditional statement, cannot be done until after the conditional statement has been completed.

$\sim$	$(p$	$\to$	$q)$	$\wedge$	$\sim$	p
	T	**T**	T		F	T
	T	**F**	F		F	T
	F	**T**	T		T	F
	F	**T**	F		T	F
	1	**3**	1		2	1

<u>Step 3:</u> Find the truth values for the conditional statement.

$\sim$	$(p$	$\to$	$q)$	$\wedge$	$\sim$	p
F	T	T	T		F	T
T	T	F	F		F	T
F	F	T	T		T	F
F	F	T	F		T	F
4	1	3	1		2	1

<u>Step 4:</u> Find the truth values for the negation of the conditional statement.

$\sim (p \rightarrow q) \wedge \sim p$

F	T	T	T	**F**	F	T
T	T	F	F	**F**	F	T
F	F	T	T	**F**	T	F
F	F	T	F	**F**	T	F
4	1	3	1	**5**	2	1

Step 5: Find the truth value of the final conjunction. Notice that this statement always has a truth value of *false*.

Exercise 7

Construct a truth table for the following statement:

$$\left[(p \rightarrow q) \wedge \sim q \right] \vee \sim p$$

Reference:
Set Theory – An Intuitive Approach
by You-Feng Lin and Shwu-Yeng T. Lin
Houghton Mifflin, 1974

We wish to thank Dr. Kenneth E. Hummel of the Trinity University Department of Mathematics for providing this reference.

V. TAUTOLOGIES AND CONTRADICITONS:

Definition 11 A statement is a ***tautology*** if it is always true. If a statement is always true, then it is a tautology.

Definition 12 A statement is a ***contradiction*** if it is always false. If a statement is always false, then it is a contradiction. Example 16 shows one statement that is a contradiction.

Example 17 Given statements *p* and *q*:

1. $p \vee \sim p$ is a tautology.

p	$\sim p$	$p \vee \sim p$
T	F	**T**
F	T	**T**

2. $q \wedge \sim q$ is a contradiction.

q	$\sim q$	$q \wedge \sim q$
T	F	**F**
F	T	**F**

3. $p \rightarrow \sim p$ is neither a tautology nor a contradiction.

p	$\sim p$	$p \rightarrow \sim p$
T	F	**F**
F	T	**T**

4. $\left[(p \to q) \wedge p\right] \to q$ is a tautology.

$\left[(p \quad \to \quad q) \quad \wedge \quad p\right] \quad \to \quad q$

T	T	T	T	T	**T**	T
T	F	F	F	T	**T**	F
F	T	T	F	F	**T**	T
F	T	F	F	F	**T**	F
1	2	1	3	1	**4**	1

Exercise 8

i. Using a truth table, prove: $\left[(p \to q) \wedge \sim q\right] \to \sim p$ is a tautology.

ii. Using a truth table, prove: $\left[(p \to q) \wedge \sim p\right] \to \sim q$ is not a tautology.

VI. IMPLICATION

Definition 13a Let p and q be any two statements. p ***implies*** q if and only if $p \vee q \equiv q$.

In other words, p implies q if the statement "p or q" is logically equivalent to the statement q.

Notation: $p \Rightarrow q$ (Read "p implies q")

$p \not\Rightarrow q$ (Read "p does not imply q")

Example 18a Let p be the statement "*The animal is a beagle.*" Let q be the statement "*The animal is a dog.*" We can say that $p \Rightarrow q$ since the statement "*The animal is a beagle or the animal is a dog*" is logically equivalent to the statement "*The animal is a dog*."

Note: If $p \Rightarrow q$, we say that q *follows* from p, or that q is *derived from* p. $p \Rightarrow q$ is a relation between statements p and q. This relation is called an implication. It should not be confused with the conditional $p \rightarrow q$, which is a new statement made from statements p and q.

Example 18b Prove this relationship:

$$\left[(p \rightarrow q) \wedge \sim q\right] \Rightarrow \sim p$$

To prove this, we need to show:

$$\left[(p \rightarrow q) \wedge \sim q\right] \vee \sim p \equiv \sim p$$

$[(p$	$\rightarrow$	$q)$	$\wedge$	$\sim$	$q]$	$\vee$	$\sim$	p
T	T	T	F	F	T	**F**	**F**	T
T	F	F	F	T	F	**F**	**F**	T
F	T	T	F	F	T	**T**	**T**	F
F	T	F	T	T	F	**T**	**T**	F
1	3	1	4	2	1	**5**	**2**	1

The above truth table shows that $\left[(p \rightarrow q) \wedge \sim q\right] \vee \sim p$ has the same truth value as $\sim p$. Therefore, it has been shown that $\left[(p \rightarrow q) \wedge \sim q\right] \vee \sim p \equiv \sim p$. Thus, it can be said that $\left[(p \rightarrow q) \wedge \sim q\right] \vee \sim p \Rightarrow \sim p$.

<u>Definition 13b</u> As a consequence of Definition 13a, a second way to define implication can be given: In section I G, the following theorem was proven: $\boldsymbol{p \rightarrow q}$ **is true if and only if** $\boldsymbol{p \vee q \equiv q}$. We can use that theorem to make the following statement:

$$\boldsymbol{p \Rightarrow q} \textbf{ if and only if } \boldsymbol{p \rightarrow q} \textbf{ is a tautology}$$

<u>Proof of statement:</u>

1. If $p \Rightarrow q$, then $p \vee q \equiv q$ (Definition 13a). Thus, from the theorem above, $p \rightarrow q$ is always a true statement (and is therefore a tautology).

2. Likewise, if $p \rightarrow q$ is a tautology, then $p \vee q \equiv q$. Thus, $p \Rightarrow q$.

p	q	$p \vee q$	$p \rightarrow q$	
T	T	T	T	
T	F	T	F	←This row fails to meet the criteria of the theorem and is therefore removed for the proof.
F	T	T	T	
F	F	F	T	

Exercise 9

i. Use a truth table to prove $\left[(p \rightarrow q) \wedge p\right] \Rightarrow q$

ii. Using Definition 13a of implication, prove If $p \Rightarrow q$ and $q \Rightarrow r$, then $p \Rightarrow r$

<u>VII. THEOREMS OF LOGIC</u>

A. THEOREM 1 ***The Algebra of Statements (Propositional Calculus)***. Let p, q, and r be any three statements.

a. ***Closure***: $p \vee q$ is a statement and $p \wedge q$ is a statement.

b. ***Commutative Properties***: $p \vee q \equiv q \vee p$ and $p \wedge q \equiv q \wedge p$

c. ***Associative Properties:*** $(p \vee q) \vee r \equiv p \vee (q \vee r)$ and $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$

d. ***Identity Properties:*** If c is a contradiction and t is a tautology, then $p \vee c \equiv p$ and $p \wedge t \equiv p$.

e. ***Inverse Elements***: $p \vee \sim p \equiv t$ and $p \wedge \sim p \equiv c$

f. ***Distributive Properties:*** $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$ and $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$

g. ***DeMorgan's Properties:*** $\sim (p \vee q) \equiv \sim p \wedge \sim q$ and $\sim (p \wedge q) \equiv \sim p \vee \sim q$

h. ***Idempotent Properties:*** $p \vee p \equiv p$ and $p \wedge p \equiv p$

B. THEOREM 2 ***Reflexive Property*** For any statement p, $p \Rightarrow p$.

C. THEOREM 3 ***Antisymmetric Property*** If $p \Rightarrow q$ and $q \Rightarrow p$, then $p \equiv q$.

D. THEOREM 4 ***Transitive Property*** If $p \Rightarrow q$ and $q \Rightarrow r$, then $p \Rightarrow r$.

E. THEOREM 5 ***Rule of Modus Ponens*** $\left[(p \to q) \wedge p\right] \Rightarrow q$

F. THEOREM 6 ***Rule of Modus Tollens*** $\left[(p \to q) \wedge \sim q\right] \Rightarrow \sim p$

G. THEOREM 7 ***Hypothetical Syllogism*** $\left[(p \to q) \wedge (q \to r)\right] \Rightarrow (p \to r)$

H. THEOREM 8

a. $p \wedge q \Rightarrow p$

b. $p \Rightarrow p \vee q$

I. THEOREM 9 $p \Rightarrow q$ if and only if $p \to q$ is a tautology. This means that if the statement $p \to q$ is always true, then $p \Rightarrow q$. Also, if $p \Rightarrow q$, then $p \to q$ is a tautology. (This was proven in section VI)

<u>VIII. ARGUMENTS</u>

A. DEFINITION

<u>**Definition 14**</u> An ***argument*** is the assertion that the conjunction of two or more statements, called the ***premises***, implies another statement, called the ***conclusion***. The argument will say that the ***conclusion*** is ***derived*** from the ***premises***..

B. PROOF OF VALIDITY

Definition 15a An argument is ***valid*** if and only if the conjunction of the premises implies the conclusion. The conjunction of the premises is also called the ***hypothesis***. Remember that for the hypothesis to *imply* the conclusion, this relationship must hold: $(\textbf{hypothesis}) \vee (\textbf{conclusion}) \equiv (\textbf{conclusion})$

Definition 15b Another way to define implication was shown in Definition 14b. Using this alternate definition of implication, it can be said that an argument is valid if and only if $(\textbf{hypothesis}) \to (\textbf{conclusion})$ **is a tautology**

This alternate definition can be used to show that an argument is not valid. If it can be shown that there is at least one case where $(\text{hypothesis}) \to (\text{conclusion})$ is false, then the argument is not valid.

Example 19a The following argument is valid:

Theorem 5: $\left[(p \to q) \wedge p\right] \Rightarrow q$.

Proof of Theorem 5:

The truth table was constructed in exercise 9 i. The final truth values are shown below.

p	q	$\left[(p \to q) \wedge p\right] \vee q$
T	**T**	**T**
T	**F**	**F**
F	**T**	**T**
F	**F**	**F**

Since q and $\left[(p \to q) \wedge p\right] \vee q$ have identical truth tables, they are logically equivalent statements. Therefore, since $\left[(p \to q) \wedge p\right] \vee q \equiv q$, it can be stated that $\left[(p \to q) \wedge p\right] \Rightarrow q$.

Thus, the argument is valid.

Definition 16b can also be used to prove the validity of the argument:

$[(p$	$\to$	$q)$	$\wedge$	$p]$	$\to$	q
T	T	T	T	T	**T**	T
T	F	F	F	T	**T**	F
F	T	T	F	F	**T**	T
F	T	F	F	F	**T**	F
1	2	1	3	1	**4**	1

Since $\left[(p \to q) \wedge p\right] \to q$ is a tautology, $\left[(p \to q) \wedge p\right] \Rightarrow q$. Thus, the argument is valid.

Example 19b The following argument is not valid:

$$\left[(p \to q) \wedge q\right] \Rightarrow p$$

Proof:

p	q	$\left[(p \to q) \wedge q\right] \vee p$
T	T	T
T	F	T
F	T	**T**
F	F	F

As you can see, $\left[(p \to q) \wedge q\right] \vee p$ does not have exactly the same truth values as p. Therefore, $\left[(p \to q) \wedge q\right] \not\equiv p$.

Thus, the argument is not valid.

Using Definition 16b, the proof can also be shown this way:

$$\left[(p \quad \to \quad q) \quad \wedge \quad q\right] \quad \to \quad p$$

T	T	T	T	T	**T**	T
T	F	F	F	F	**T**	T
F	T	T	F	T	**F**	F
F	T	F	F	F	**T**	F
1	2	1	3	1	**4**	1

Since $\left[(p \to q) \wedge q\right] \to p$ is not a tautology, $\left[(p \to q) \wedge q\right] \not\Rightarrow p$. Thus, the argument is not valid.

To understand the above example, consider the following: We could say *"If the animal is a beagle, the it is a dog."* However, the compound statement *"If the animal is a beagle, then it is a dog: <u>and</u> the animal is a dog"* does not imply that the animal is a beagle.

NOTE: If the hypothesis implies the conclusion, then the argument is valid. However this does <u>not</u> mean that the conclusion is true. For instance, Example 19a presented an argument that was valid. However, statement q, the conclusion of the argument, could have a truth value of false. Validity only tests the correctness of the argument, <u>not</u> the truth value of the conclusion.

Likewise, an argument can be invalid, but have a conclusion that is true.

Exercise 10

Prove the following argument is valid: (hint: see example 15)

$$\left[(p \to q) \wedge (q \to r)\right] \Rightarrow (p \to r)$$

What Theorem did you prove? ____

____

C. SYMBOLIC REPRESENTATION OF ARGUMENTS:

Example 20 Check the validity of the following argument:

> If $2+2=5$, then 12 is a negative integer.
> $2+2=5$.
> Therefore, 12 is a negative integer.

Let *p* be the statement "$2+2=5$." Let *q* be the statement "12 is a negative integer."

The ***symbolic form*** of the argument above looks like this:

$$\begin{array}{l} P_1 : p \to q \\ \underline{P_2 : p \quad\quad} \\ C : q \end{array}$$

P_1 is the first premise, the statement "If $2+2=5$, then 12 is a negative integer."

P_2 is the second premise, the statement "$2+2=5$."

The argument states that the conjunction of the premises implies the conclusion, or $\left[(p \to q) \wedge p\right] \Rightarrow q$. This argument is valid by the Rule of Modus Ponens, shown in example 19b.

Example 21 Check the validity of the following argument:

> If $2\times 6=12$, then 23 is a positive integer.
> 23 is a positive integer.
> Therefore, $2\times 6=12$.

Let *p* be the statement "$2\times 6=12$." Let *q* be the statement "23 is a positive integer."
The ***symbolic form*** of the argument looks like this:

$P_1 : p \rightarrow q$
$\underline{P_2 : q \qquad}$
$C : p$

This argument states that $\left[(p \rightarrow q) \wedge q\right] \Rightarrow p$.
In example 19c, it was shown that this argument is not valid.

Exercise 11

Use symbolic representation to check the validity of the following argument:

If it rains, then the soccer game is delayed.
It is not raining.
Therefore, the soccer game is not delayed.

IX. CONJUNCTIONS OF PREMISES AND CONCLUSIONS:

The arguments in section VIII had only two premises given. The validity of this type of argument is easily shown by applying one of the Theorems of Logic from section VII to the conjunction of the two premises.

Some arguments contain several premises. To show the validity of these arguments, it is necessary to do the proof in several steps. A reason must be given for each step taken in a proof. Other than the Theorems of Logic, there are other rules that may also be used when providing the validity of an argument:

1. The conjunction of two premises follows from the premises.
2. The conjunction of a premise and the conclusion of a valid argument follows.
3. The conjunction of two conclusions of valid arguments follows from the premises.

Note: When checking the validity of an argument, carefully list each step and give the reason that the step can be taken. If the argument cannot be proven to be valid, it should be shown invalid. This can be done by showing that $(\text{hypothesis}) \rightarrow (\text{conclusion})$ is not a tautology. It is sufficient to show that there is at least one case here $(\text{hypothesis}) \rightarrow (\text{conclusion})$ has a truth value of false.

Example 22 Check the validity of the following argument:

If it is raining, the picnic is cancelled. If it is not raining, the soccer game is not delayed. But the picnic is not cancelled. Therefore, the soccer game is not delayed.

Let r denote "It is raining."

Let p denote "The picnic is cancelled."

Let s denote "The soccer game is delayed."

The symbolic form of the argument is:

$$\begin{array}{l} P_1: \quad r \to p \\ P_2: \sim r \to \sim s \\ \underline{P_3: \sim p \qquad} \\ C: \quad \sim s \end{array}$$

Proof:

1.	$r \to p$	Premise (given)
2.	$\sim r \to \sim s$	Premise (given)
3.	$\sim p$	Premise (given)
4.	$(r \to p) \wedge \sim p$	Conjunction of premises 1 and 3
5.	$\sim r$	Rule of Modus Tollens applied to step 4
6.	$(\sim r \to \sim s) \wedge \sim r$	Conjunction of premise 2 and valid argument 5
7.	$\sim s$	Rule of Modus Ponens applied to step 6

Therefore, the argument is valid. $\therefore$ Valid

Example 23 Check the validity of the following argument:

If an object is less dense that water (l), it will float (f). The water displaced by an object will weigh more than the object (m), or the object will not float. But the water displaced by the object does not weigh more than the object. Thus, the object is not less dense than water.

The symbolic form of the argument is:

$$P_1: \quad l \to f$$
$$P_2: m \vee \sim f$$
$$P_3: \sim m$$
$$C: \quad \sim l$$

Proof:

1.	$l \to f$	Premise (given)
2.	$m \vee \sim f$	Premise (given)
3.	$\sim m$	Premise (given)
4.	$\sim f \vee m$	Commutative Property applied to step 2
5.	$f \to m$	Property of Logical Equivalence (Example 12.4)
6.	$(f \to m) \wedge \sim m$	Conjunction of valid argument 5 and premise 3
7.	$\sim f$	Rule of Modus Tollens applied to step 6
8.	$(l \to f) \wedge \sim f$	Conjunction of premise 1 and valid argument 7
9.	$\sim l$	Rule of Modus Tollens applied to step 8

Therefore, the argument is valid. $\therefore$ Valid

Example 24 Check the validity of the following argument:

If the sun has set (s), then the street lamps are on (l). Either the electrical circuits are working (e), or the street lamps are not on. the electrical circuits are not working. Therefore, the sun has set.

The symbolic form of the argument is:

$$P_1: \quad s \to l$$
$$P_2: e \vee \sim l$$
$$P_3: \sim e$$
$$C: \quad s$$

Trying to prove that this argument is valid using the method above will prove to be futile. Indeed, it seems intuitive that this argument is not valid. However, the argument must be shown to be invalid using the definition of validity. Recall that Definition 16b stated that $(\text{hypothesis}) \to (\text{conclusion})$ must be a tautology for a valid argument. A truth table for $(\text{hypothesis}) \to (\text{conclusion})$ is shown below.

$[(s$	$\to$	$l)$	$\wedge$	$(e$	$\vee$	$\sim$	$l)$	$\wedge$	$\sim$	$e]$	$\to$	s
T	T	T	T	T	T	F	T	F	F	T	**T**	T
T	T	T	F	F	F	F	T	T	T	F	**T**	T
T	F	F	F	T	T	T	F	F	F	T	**T**	T
T	F	F	F	F	T	T	F	F	T	F	**T**	T
F	T	T	T	T	T	F	T	F	F	T	**T**	F
F	T	T	F	F	F	F	T	F	T	F	**T**	F
F	T	F	T	T	T	T	F	F	F	T	**F**	F
F	T	F	T	F	T	T	F	T	T	F	**F**	F
1	3	1	4	1	3	2	1	5	2	1	**6**	1

Since $[(s \to l) \wedge (e \vee \sim l) \wedge \sim e] \to s$ is not a tautology, $[(s \to l) \wedge (e \vee \sim l) \wedge \sim e] \not\Rightarrow s$. Thus, the argument is not valid.

Note: It is not necessary to show the entire truth table. If it can be shown that there is at least one case where $(\text{hypothesis}) \to (\text{conclusion})$ is false, then the argument has been shown to be invalid.

Example 25 Check the validity of the following argument.

If the Empire State Building is tall (t), then it will be struck by lightning (l). If the Empire State Building is not tall, it will not touch the clouds (c). The Empire State Building is not struck by lightning. Therefore, the Empire State Building is not tall.

The symbolic form of the argument is:

$$P_1:\ t \to l$$
$$P_2: \sim t \to \sim c$$
$$\underline{P_3: \sim l\qquad}$$
$$C:\ \sim t$$

This argument has the same form as Example 22. The argument was shown to be valid. In this case, we know that the conclusion (the Empire State Building is not tall) is false. However, the argument is logically valid.

Example 26 Check the validity of the following argument:

If the ocean is deep (d), then the log will float (f). The log floats. Therefore, the ocean is deep.

The symbolic form of the argument is:

$$\begin{array}{l} P_1: \quad d \rightarrow l \\ \underline{P_2: \quad l \qquad} \\ C: \quad d \end{array}$$

It is clear that the conclusion of this argument (the ocean is deep) is a true statement. However, this argument has the same form as Example 19c. In that example, it was shown that the argument is not logically valid.

Note: The above two examples illustrate the point that the conclusion of a valid argument is not necessarily true, and the conclusion of an invalid argument might possibly be true.

Exercise 12

Rewrite the argument in symbolic form using the suggested letters. Then check the validity of the argument.

i. The space shuttle landed in Florida (f), or it landed in California (c). If the space shuttle landed in California, then the alternate landing schedule was used (a). But, the alternate landing schedule was not used. Therefore, the space shuttle landed in Florida.

ii. If February has 29 days (d), then it is a leap year (l). If it is a leap year, then there will be a presidential election (e). But, there is not a presidential election. Therefore, February does not have 29 days.

X. QUANTIFIERS AND QUANTIFIED STATEMENTS:

A. DEFINITIONS AND EXAMPLES

There are two basic types of quantified statements. The first type says something about every member of a group or class of things. For example, "*All dogs are hounds*." The second type of quantified statement says that a group with a certain characteristic has at least one member. For example, "*Some dogs are wild*" or "*There is a dog that is a pointer.*" Quantified statements can be combinations of these two types. For example, "*All dogs are not hounds and some dogs are wild*."

The adjective "***All***" is called a ***universal quantifier***.

The adjective "***Some***" is called an ***existential quantifier***.

The universal quantifier "*All* "is sometimes replaced with "*For all*," "*For every,*" or "*For each.*" Each of the following statements is saying the same thing. In other words, each statement would have the same truth value.

1. All dogs are hounds.
2. For all dogs x, x is a hound.
3. For every dog x, x is a hound.
4. For each dog x, x is a hound.
5. For all x, if x is a dog, then x is a hound.

Statements (2), (3), (4), and (5) may seem awkward, but there are advantages to each form.

The existential quantifier "*Some* " is sometimes replaced with "*There is,* " "*There exists,*" or "*There is at least one* ." Each of the following statements is saying the same thing. In other words, each statement would have the same truth value.

1. Some dogs are pointer.
2. There is a dog that is a pointer.

3. There exists a dog that is a pointer.
4. There is at least one dog that is a pointer.
5. There is an x such that x is a dog and x is a pointer.

Remark: It is customary to <u>not</u> read plural into the quantifier "*Some* " even though statement (1) above seems to suggest that more than one dog is a pointer. We will follow the convention of not assuming that "*Some* " indicates more than one.

The phrase "*For some* " should be avoided since it is not always clear whether or not the intended meaning is "*All* " or "*Some.*"

B. TERMS AND PREDICATES:

Consider the statements:

3 is a whole number.

x is a whole number.

In these examples, "*is a whole number* " is called a ***predicate***. 3 is called a ***term***. x is also a term.

Let *W* denote the predicate "*is a whole number* ". Thus, *W3* is read as "*3 is a whole number.* " *Wx* is read as "*x is a whole number.* " We write "*W3* " rather than "*3W*."

Using the same notation, the statement " $\frac{1}{2}$ is not a whole number" is written as $\sim W\frac{1}{2}$.

The statement "*y is not a whole number* " is written *~Wy*.

C. SYMBOLIC REPRESENTATION OF QUANTIFIED STATEMENTS:

Statements without quantifiers were previously translated into symbolic form. The quantified statements can also be put in symbolic form. But first, the universal quantifier "*For all x* " and the existential quantifier "*There is an x* " must be replaced with symbols..

"$(\forall x)$" will replace and be read as "***For all x***," or "***For every x.*** "

"$(\exists x)$" will replace and be read as "***There is an x*** " or "***There exists and x***. "

Consider the statement: *All tigers are cats.*

Let T be the predicate "is a tiger."
Let C be the predicate "is a cat."

You can use the symbols to rewrite the statement in the following ways:

"For all *x*, if *x* is a tiger, then *x* is a cat," or

"For all *x*, $(Tx \rightarrow Cx)$," or

"$(\forall x)(Tx \rightarrow Cx)$,"

where "$(\forall x)$" means "For all *x*." The last statement is called the ***symbolic form*** of the original statement.

Consider the statement "*Some quadrilaterals are squares*." This statement is logically equivalent to:

"*There exists an x such that x is a quadrilateral and x is a square*."

Let Q be the predicate "*is a quadrilateral.*"

Let S be the predicate "*is a square*."

The statement can then be written symbolically as:

"There exists an *x* such that $Qx \wedge Sx$," or

"$(\exists x)(Qx \wedge Sx)$"

Note: The two statements "*All dogs are hounds* " and "*Some dogs are hounds* " appear very similar in structure. In fact, only one word of each statement is different. The difference between the statements can be seen more clearly when the statements are expressed in symbolic form:

"*All dogs are hounds* " is expressed in symbolic form: $(\forall x)(Dx \to Hx)$

"*Some dogs are hounds* " is expressed in symbolic form: $(\exists x)(Dx \wedge Hx)$

Notice that the first statement contains a conditional statement and the second statement contains a conjunction. You might ask why the first statement does not have a conjunction as well. But $(\forall x)(Dx \wedge Hx)$ translates into the absurd statement "*Everything is both a dog and a hound*."

<u>Example 27</u> Rewrite each of the following quantified statements in symbolic form.

1. Some books are not novels.
2. Not all apples are red.
3. All diamonds have brilliance.
4. Some cereals contain vitamins.

1. **Some books are not novels.** This statement can be rewritten:

There exists and x such that x is a book and x is not a novel.

Let B be the predicate "*is a book.* "
Let N be the predicate "*is a novel.* "

Symbolically this statement is written $(\exists x)(Bx \wedge \sim Nx)$.

2. **Not all apples are red.** This statement can be rewritten:

There exists an x such that x is an apple and x is not red.

Let A be the predicate "*is an apple.*"
Let R be the predicate "*is red*."

Symbolically, this statement is written $(\exists x)(Ax \wedge \sim Rx)$.

3. **All diamonds have brilliance.** The statement can also be written:

All diamonds are brilliant objects.

Therefore, you can rewrite the statement in the form:
For all x, if x is a diamond, then x is a brilliant object.

Let D be the predicate "*is a diamond*. "
Let B be the predicate "*is a brilliant object*. "

Symbolically, this statement is written $(\forall x)(Dx \rightarrow Bx)$.

4. **Some tortillas are made of flour.** This statement can be written:

Some tortillas are products which are made of flour.

Therefore, you can rewrite the statement this way:

There exists an x such that x is a tortilla and x is a product which is made of flour.

Let T be the predicate "*is a tortilla*. "
Let F be the predicate "*is a product which is made of flour*. "

Symbolically, the statement is written $(\exists x)(Tx \wedge Fx)$

D. NEGATION OF QUANTIFIED STATEMENTS:

Negation of quantified statements must satisfy the following conditions:

1. The truth value of the negation must be opposite the truth value of the original statement.
2. The form of the negation must hold for all terms and for all predicates.

<u>Example 28</u>

1. Consider the statement p: ***All cats are tigers.*** One of the following statements is its negation:

a_1 : *All cats are not tigers.*

b_1 : *Some cats are tigers.*

c_1 : *Some cats are not tigers.*

Now, we know that p is false. Also, a_1 is false while b_1 and c_1 are true. Therefore a_1 cannot be the negation of p.

2. Consider the statement q: ***All oboes are musical instruments***. One of the following statements is its negation:

a_2 : *All oboes are not musical instruments.*

b_2 : *Some oboes are musical instruments.*

c_2 : *Some oboes are not musical instruments.*

Now we know that q is true. Also, b_2 is true while a_2 and c_2 are false. Thus, b_2 cannot be the negation of q.

Statement p and statement q have the same form. Therefore, the negations of p and q must have the same form. Only the third statement in the list of possible negations can be used for both p and q. Thus, the negation of "*All cats are tigers* " is c_1, "***Some cats are not tigers***. " Likewise, the negation of "*All oboes are musical instruments* " is c_2, "***Some oboes are not musical instruments***. "

3. Using what we learned above, the negation of the statement "*All books are novels* " is the statement "*Some books are not novels*. "

Remark: When working with a *quantified statement*, adding or removing a "*not* " does not produce the negation of the statement. Removing or adding a "*not* " following the verb does give a new statement, but this statement is not necessarily the negation when the original statement is quantified with "*All* " or "*Some* ". In the next section, we will develop a method to find the negation of any quantified statement.

DETERMINATION OF NEGATIONS OF QUANTIFIED STATEMENTS THROUGH SYMBOLIC ANALYSIS:

Consider again the statements

p: *All tigers are cats.*

w: *Some tigers are not cats.*

From the above section, $w \equiv \sim p$.

Let T be the predicate "*is a tiger* " and let C be the predicate "*is a cat.* "

p can be written $(\forall x)(Tx \to Cx)$

w can be written $(\exists x)(Tx \wedge \sim Cx)$.

Thus, $(\exists x)(Tx \wedge \sim Cx)$ is the same as *w*, which is the same as *~p*

which is the same as $\sim(\forall x)(Tx \rightarrow Cx)$

which is the same as $\sim(\forall x)(\sim Tx \vee Cx)$

which is the same as $\sim(\forall x)\sim(Tx \wedge \sim Cx)$.

Therefore, you can say that $(\exists x)$ and $\sim(\forall x)\sim$ mean the same thing.

Therefore, $(\exists x) \equiv \sim(\forall x)\sim$. Hence, $\sim(\exists x) \equiv \sim\sim(\forall x)\sim \equiv (\forall x)\sim$, and $\sim(\exists x)\sim \equiv \sim\sim(\forall x) \equiv (\forall x)$. Knowing these relationships, you can use them to determine the negations of different quantified statements.

<u>Example 29</u> Using symbolic analysis, find the negation of *p: Some books are novels.* Let B be the predicate "*is a book* " and N be the predicate "*is a novel.* " Symbolically, p is written $(\exists x)(Bx \wedge Nx)$.

Thus, $\sim p \equiv \sim(\exists x)(Bx \wedge Nx)$

$\equiv (\forall x)\sim(Bx \wedge Nx)$ (using the relationship shown above)

$\equiv (\forall x)(\sim Bx \vee \sim Nx)$

$\equiv (\forall x)(Bx \rightarrow \sim Nx)$

Hence, $\sim p \equiv (\forall x)(Bx \rightarrow \sim Nx)$.

Therefore, *~p* is the statement "*All books are not novels.* "

<u>Example 30</u> Using symbolic analysis, find the negation of *q: Some apples are not red.* Let A be the predicate "*is an apple* " and R be the predicate "*is red* . " Then, $q \equiv (\exists x)(Ax \wedge \sim Rx)$.

Thus, $\sim q \equiv \sim(\exists x)(Ax \wedge \sim Rx)$

$\equiv (\forall x)\sim(Ax \wedge \sim Rx)$

$\equiv (\forall x)(\sim Ax \vee Rx)$

$\equiv (\forall x)(Ax \rightarrow Rx)$.

Hence, $\sim q \equiv (\forall x)(Ax \rightarrow Rx)$. Therefore *~q* is the statement "*All apples are red.* "

<u>Example 31</u> Using symbolic reasoning, find the negation of *r: Not all recordings are digital*. The statement *r* can also be written "*Some recordings are digital*." Let R be the predicate "*is a recording* " and D be the predicate "*is digital*. " Then, $r \equiv (\exists x)(Rx \wedge \sim Dx)$

Thus, $\sim r \quad \equiv \sim (\exists x)(Rx \wedge \sim Dx)$

$$\equiv (\forall x)(\sim Rx \vee Dx)$$

$$\equiv (\forall x)(Rx \rightarrow Dx)$$

Hence, $\sim r \equiv (\forall x)(Rx \rightarrow Dx)$. Therefore, $\sim r$ is the statement "*All recordings are digital.*"

Example 32 Using symbolic analysis, find the negation of s: *All lemons are not sweet.* Let L be the predicate "*is a lemon*" and S be the predicate "*is sweet*." Then, $s \equiv (\forall x)(Lx \rightarrow \sim Sx)$.

Thus, $\sim s \quad \equiv \sim (\forall x)(Lx \rightarrow \sim Sx)$

$$\equiv (\exists x) \sim (Lx \rightarrow \sim Sx)$$

$$\equiv (\exists x) \sim (\sim Lx \vee \sim Sx)$$

$$\equiv (\exists x)(Lx \wedge Sx)$$

Hence, $\sim s \equiv (\exists x)(Lx \wedge Sx)$. Therefore, $\sim s$ is the statement "*Some lemons are sweet.*"

SUMMARY OF NEGATIONS OF QUANTIFIED STATEMENTS

p	$\sim p$
All x are y.	Some x are not y. (Not all x are y.)
Some x are y.	All x are not y. (No x is a y.)
Some x are not y. (Not all x are y.)	All x are y.
All x are not y. (No x is a y.)	Some x are y.

E. COMPOUND QUANTIFIED STATEMENTS:

If a statement is a compound statement that also contains quantifiers, then the rules from section III must be used with the rules above to determine the negation.

Example 33 Find the negation of the following:

All dogs are beagles and all cats are tigers.

If p is the statement "*all dogs are beagles*" and q is the statements "*all cats are tigers*", DeMorgan's Law tells us that $\sim (p \wedge q) \equiv \sim p \vee \sim q$. We also know that we can use the chart above to determine $\sim p$ and $\sim q$.

Thus, the negation of "*All dogs are beagles and all cats are tigers*" would be

Some dogs are not beagles <u>or</u> some cats are not tigers.

<u>Example 34</u> Find the negation of the following:

All dogs are beagles or some frogs are not amphibians.

Let p be the statement "*all dogs are beagles*" and q be the statement "*some frogs are not amphibians.*" Using DeMorgan's Law, we know that $\sim(p \vee q) \equiv \sim p \wedge \sim q$. We can use the chart above to find ~*p* and ~*q*. Thus the negation of "*All dogs are beagles or some frogs are not amphibians*" would be

Some dogs are not beagles <u>and</u> all frogs are amphibians.

<u>Example 35</u> Find the negation of the following:

If some dogs are beagles, then all cats are kittens.

Let p be the statement "*some dogs are beagles*" and q be the statement "*all cats are kittens.*" We know from example 12.9 that $\sim(p \rightarrow q) \equiv p \wedge \sim q$. Thus, the negation of "*If some dogs are beagles, then all cats are kittens*" would be

Some dogs are beagles <u>and</u> some cats are not kittens.

F. RULE OF INSTANTIATION OR TERM SUBSTITUTION:

Consider the following argument:

All cats have whiskers.
<u>Garfield is a cat. .</u>
Therefore, Garfield has whiskers.

It appears that this argument is valid. To prove it is valid, the ***Rule of Instantiation*** or ***Term Substitution*** is used. Namely, in any statement preceded by a *universal quantifier*, the variable indicated by the quantifier may be replaced throughout the statement by another term. In this case, we have substituted the term Garfield for the term Cat. We can do this because of the use of the universal quantifier "all."

In the above argument, let C be the predicate "*is a cat*" and W be the predicate "*has whiskers* ."

The first premise is symbolized by $(\forall x)(Cx \rightarrow Wx)$.
The second premise is symbolized by C_{Garfield}
The conclusion is symbolized by W_{Garfield}
The symbolic form of the argument is:

$$\begin{array}{l} P_1: (\forall x)(Cx \rightarrow Wx) \\ \underline{P_2: C_{\text{Garfield}}} \\ C: W_{\text{Garfield}} \end{array}$$

Below is a proof for this argument

Proof:

1.	$(\forall x)(Cx \to Wx)$	Given
2.	C_{Garfield}	Given
3.	$C_{\text{Garfield}} \to W_{\text{Garfield}}$	Instantiation (I)
4.	$(C_{\text{Garfield}} \to W_{\text{Garfield}}) \wedge C_{\text{Garfield}}$	Conjunction (3 and 2)
5.	W_{Garfield}	Modus Ponens (MP4)

$\therefore$ Valid

Example 36 Prove the following argument is valid.

All camellias are white.
The bluebonnet is not white.
Therefore, the bluebonnet is not a camellia.

Let C be the predicate "*is a camellia*" and W be the predicate "*is white.*"

The symbolic form of the argument is:

$$\begin{array}{l} P_1 : (\forall x)(Cx \to Wx) \\ \underline{P_2 : \sim W_{\text{bluebonnet}}} \\ C : \sim C_{\text{bluebonnet}} \end{array}$$

Proof:

1.	$(\forall x)(Cx \to Wx)$	Given
2.	$\sim W_{\text{bluebonnet}}$	Given
3.	$C_{\text{bluebonnet}} \to W_{\text{bluebonnet}}$	Instantiation (I)
4.	$(C_{\text{bluebonnet}} \to W_{\text{bluebonnet}}) \wedge \sim W_{\text{bluebonnet}}$	Conjunction (3 and 2)
5.	$\sim C_{\text{bluebonnet}}$	Modus Tollens (MT4)

$\therefore$ Valid

Exercise 13

i. Given the statement: *All turtles have shells.*

Let T be the predicate "*is a turtle.*" Let S be the predicate "*is an animal with a shell.*"

a. Write the statement in symbolic form.

b. Write its negation symbolically.

c. Write the negation in words.

ii. Let p be the statement: *Some birds are eagles.*

Let B be the predicate "*is a bird* ", and let E be the predicate "*is an eagle* ."

a. Write p symbolically.

b. Using symbolic analysis, find the negation of p.

c. Translate the negation (write it in words).

iii. Let q be the statement: *Not all books are novels.*

Let B be the predicate "*is a book* " and N be the predicate "*is a novel.* "

a. Write q symbolically.

b. Using symbolic analysis, find the negation of q.

c. Translate the negation.

iv. Write the negation of the following statement:

Some birds are eagles or not all books are novels.

v. Write the symbolic form for the following argument.

All insects have six legs.
An ant is an insect.
Therefore, an ant has six legs.

Let I be the predicate "*is an insect*" and S be the predicate "*is a creature with six legs*."

Construct a proof for the argument above.

CHAPTER 2: ELEMENTARY SET THEORY

I. BASIC CONCEPTS

1.1 **Definition** A ***set*** is a collection of objects together with some rule to determine whether a given object belongs to this collection. Any object of this collection is called an ***element***. of the set.

Notation: The *name* of a set is denoted with a capital letter – *A, B,* etc.

The *description* of the set can be given in the following ways:

1. Each element of the set is listed within a set of brackets: $\{ \quad \}$.

2. Within the brackets, the first few elements are listed, with dots following to show that the set continues with the selection of the elements following the same rule as the first few.

3. Within the brackets, the set is described by writing out the exact rule by which elements are chosen. The name given each element is separated from the selection rule with a vertical line.

1.2 **Examples**

(a) Denote by A the set of natural numbers which are greater than 25. The set could be written in the following ways:

$\{26, 27, 28, ...\}$ (using the second notation listed above)

$\{x \mid x \text{ is a natural number and } x > 25\}$ (using the third notation above)

The above description is read "the set of all x such that x is a natural number and x is greater than25."

Note that 32 is an element of A. We write $\mathbf{32 \in A}$, where "$\in$" denotes "*is an element of.*" Also, $\mathbf{6 \notin A}$, where "$\notin$" denotes "*is not an element of.* "

(b) Let B be the set of numbers $\{3,5,15,19,31,32\}$. Again the elements of the set are natural numbers, however, the rule is given by actually listing each element of the set (as in the first notation). We see that $15 \in B$, but $23 \notin B$.

(c) Let C e the set of all natural numbers which are less than 1. In this set, we observe that there are no elements. Hence, C is said to be an ***empty set***. A set with no elements is denoted by $\varnothing$.

1.3 **Definition** A set A is said to be a ***subset*** of a set B if *every* element of A is an element of B.

Notation: To indicate that set A is a subset of set B, we use the expression $\mathbf{A \subset B}$, where "$\subset$" denotes "*is a subset of.*" $\mathbf{A \not\subset B}$ means that A is *not* a subset of B.

1.4 **Examples**

(a) Let B be the set of natural numbers. Let A be the set of even natural numbers.
Clearly, A is a subset of B. However, B is not a subset of A, for $3 \in B$, but $3 \notin A$.

(b) An empty set is a subset of *any* set B. If this were not so, there would be some element $x \in \varnothing$ such that $x \notin B$. However, this would contradict with the definition of an empty set as a set with no elements.

1.5 **Theorem: Properties of Sets**

Let *A, B,* and *C* be sets.

1. For any set A, $A \subset A$ (Reflexive Property)

2. If $A \subset B$ and $B \subset C$, then $A \subset C$ (Transitive Property)

1.6 **Definition** Two sets, *A* and *B*, are said to be **equal** if and only if *A* is a subset of *B* and *B* is a subset of *A*.

To indicate that two sets, *A* and *B*, are equal, we use the symbol $\boldsymbol{A = B}$. This means that sets *A* and *B* contain *exactly the same elements*. $\boldsymbol{A \neq B}$ means that *A* and *B* are not equal sets.

1.7 **Example** Let *A* be the set of even natural numbers and *B* be the set of natural numbers which are multiples of 2. Clearly, $A \subset B$ and $B \subset A$. Therefore, since *A* and *B* contain exactly the same elements, $A = B$.

1.8 **Remarks**

(a) Two equal sets always contain the same elements. However the rules for the sets may be written differently, as in example 1.7.

(b) Since any two empty sets are equal, we will refer to any empty set as ***the*** empty set.

(c) *A* is said to be a ***proper subset*** of *B* if and only if:

(i) $A \subset B$

(ii) $A \neq B$, and

(iii) $A \neq \varnothing$.

1.9 **Theorem: Properties of Set Equality**

(a) For any set A, $A = A$. (Reflexive Property)

(b) If $A = B$, then $B = A$. (Symmetric Property)

(c) If $A = B$ and $B = C$, then $A = C$. (Transitive Property)

1.10 **Definition** Let *A* and *B* be subsets of a set *X*. The ***intersection*** of *A* and *B* is the set of all elements in *X* common to both *A* and *B*.

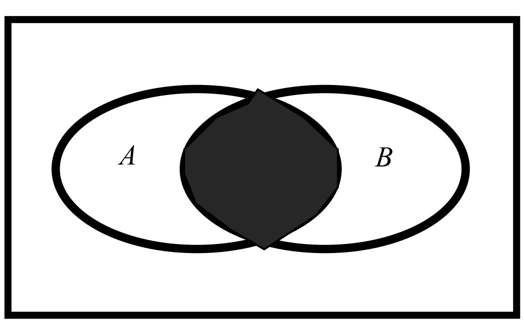
Notation: "$\boldsymbol{A \cap B}$" denotes "*A* intersection *B* " or the intersection of sets *A* and *B*.

Thus, $A \cap B = \{x \in X \mid x \in A \text{ and } x \in B\}$, or $A \cap B = \{x \in X \mid x \in A \wedge x \in B\}$.

1.11 **Examples**

(a) Given that the box below represents *X*, the shaded area represents $A \cap B$:

X:



(b) Let $A=\{2,4,5\}$ and $B=\{1,4,6,8\}$. Then, $A\cap B=\{4\}$.

Note: A set that has only one element, such as $\{4\}$, is sometimes called a singleton set.

(c) Let $A=\{2,4,5\}$ and $B=\{1,3\}$. Then $A\cap B=\varnothing$.

1.12 **Remarks**

(a) If, as in the above example 1.11c, A and B are two sets such that $A\cap B$ is the empty set, we say that A and B are ***disjoint***..

(b) Given sets A and B, $x\in A\cap B$ if and only if $x\in A$ and $x\in B$.

1.13 **Definition** Let A and B be subsets of a set X. The ***union*** of A and B is the set of all elements belonging to A or B.

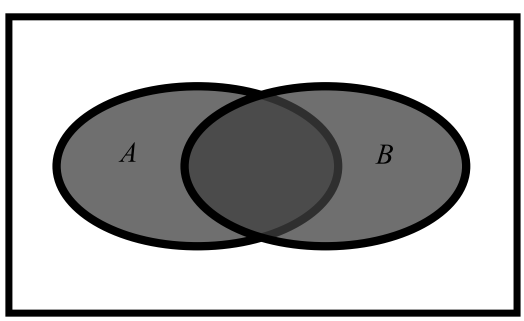
Notation: " $\boldsymbol{A\cup B}$ " denotes "A union B " or the union of sets A and B.

Thus, $A\cup B=\{x\in X\mid x\in A \text{ or } x\in B\}$, or $A\cup B=\{x\in X\mid x\in A \vee x\in B\}$.

1.14 **Examples**

(a) Given that the box below represents X, the shaded area represents $A\cup B$:

X:



(b) Let $A=\{2,4,5\}$ and $B=\{1,4,6,8\}$.

Then, $A\cup B=\{1,2,4,5,6,8\}$.

1.15 **Remark** Given sets A and B, $x\in A\cup B$ if and only if $x\in A$ or $x\in B$.

1.16 **Definition** Let A and B be subsets of a set X. The set $B-A$, called the ***difference*** of B and A, is the set of all elements in B which are not in A.

Thus, $B-A=\{x\in X \mid x\in B \text{ and } x\notin A\}$.

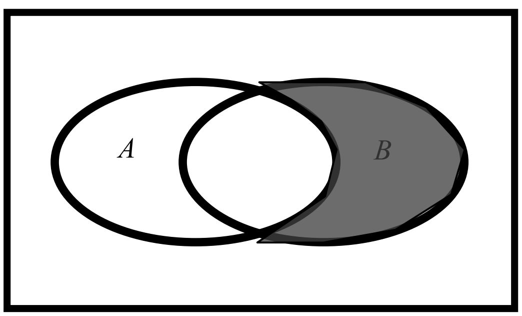
1.17 **Examples**

(a) Let $B=\{2,3,6,10,13,15\}$ and $A=\{2,10,15,21,22\}$. Then $B-A=\{3,6,13\}$.

(b) Let X be the set of natural numbers and A be the set of odd natural numbers. Then, $X-A=$ the set of even natural numbers; or $X-A=\{x \mid x \text{ is a natural number and } x \text{ is even}\}$.

(c) Given that the box below represents X, the shaded area represents $B-A$.

X:



1.18 **Definition** If $A\subset C$, then $X-A$ is sometimes called the ***complement*** of A with respect to X.

Notation: The following symbols are used to denote the complement of A with respect to X: $\complement_X A$, $\complement A$, $\sim A$, $\overline{A}$, and A'

Thus, $\complement_X A=\{x\in X \mid x\notin A\}$.

1.19 **Theorem** Let A and B be subsets of X.

Then, $A-B=A\cap \complement B$.

II. PROPERTIES OF UNION, INTERSECTION, AND COMPLEMENTATION

2.1 **Theorem** Let X be an arbitrary set and let $P(X)$ be the set of all subsets of X. $P(X)$ is called the ***power set*** of X. Let A, B, and C be arbitrary elements of $P(X)$.

(a) $A \cap B = B \cap A$ (Commutative Law for Intersection)

$A \cup B = B \cup A$ (Commutative Law for Union)

(b) $A \cap (B \cap C) = (A \cap B) \cap C$ (Associative Law for Intersection)

$A \cup (B \cup C) = (A \cup B) \cup C$ (Associative Law for Union)

(c) $A \cap B \subset A$

(d) $A \cap X = A$; $A \cup \varnothing = A$

(e) $A \subset A \cup B$

(f) $A \cup X = X$; $A \cap \varnothing = \varnothing$

(g) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ (Distributive Law of Union with respect to Intersection)

$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ (Distributive Law of Intersection with respect to Union)

(h) $\complement(A \cup B) = \complement A \cap \complement B$

$\complement(A \cap B) = \complement A \cup \complement B$

(i) $A \cap \complement A = \varnothing$; $A \cup \complement A = X$

We will prove some of the above properties and leave the remaining as exercises:

Proof of a: $A \cap B = B \cap A$

This proof can be done in two ways.

1. The *first proof* uses the fact that two sets will be equal only if $(A \cap B) \subset (B \cap A)$ <u>and</u> $(B \cap A) \subset (A \cap B)$.

(i) Let x be an element of $A \cap B$

Therefore, $x \in A \wedge x \in B$	definition of $A \cap B$
Thus, $x \in B \wedge x \in A$	Commutative property of $\wedge$
Hence, $x \in B \cap A$	definition of $B \cap A$
Therefore, $A \cap B \subset B \cap A$	

(ii) Let x be an element of $B \cap A$

Therefore, $x \in B \wedge x \in A$	definition of $B \cap A$
Thus, $x \in A \wedge x \in B$	Commutative property of $\wedge$
Hence, $x \in A \cap B$	definition of $A \cap B$
Therefore, $B \cap A \subset A \cap B$	

Thus, $A \cap B = B \cap A$

2. The *second proof* of part a uses the definition of the sets $A \cap B$ and $B \cap A$.

$A \cap B = \{x \mid x \in A \cap B\}$	
$= \{x \mid x \in A \wedge x \in B\}$	definition of $A \cap B$
$= \{x \mid x \in B \wedge x \in A\}$	Commutative property of $\wedge$
$= \{x \mid x \in B \cap A\}$	definition of $B \cap A$
$= B \cap A$	

Proof of c: $(A \cap B) \subset A$

It must be shown that each element of $A \cap B$ is an element of A.

Let $x \in A \cap B$	
Thus, $x \in A \;\wedge\; x \in B$	definition of $A \cap B$
Hence, $x \in A$	Theorem 8 of Logic Theorems ($p \wedge q \Rightarrow p$)

Therefore, $\left(A \cap B\right) \subset A$

Proof of d: $A \cap X = A$

(i)	$A \cap X \subset A$	part c above
(ii)	Let $x \in A$	
	Thus, $x \in X$	$A \subset X$ is given
	Hence, $x \in A \;\wedge\; x \in X$	definition of $\wedge$
	Therefore, $x \in A \cap X$	definition of $\cap$
	Thus, $A \subset A \cap X$	definition of $\subset$

Thus, $A \cap X = A$.

Proof of h: $\complement(A \cup B) = \complement A \cap \complement B$ **Again, this proof can be done in two ways:**

1. Proof 1 (to show that $\complement(A \cup B) = \complement A \cap \complement B$; and $\complement A \cap \complement B = \complement(A \cup B)$)

(i) Let $x \in \complement(A \cup B)$

Therefore, $x \notin (A \cup B)$	Definition of $\complement$
Thus, $\sim(x \in A \cup B)$	Definition of $\notin$
Hence, $\sim(x \in A \vee x \in B)$	Definition of $A \cup B$
Therefore, $\sim(x \in A\ ) \wedge \sim(x \in B)$	DeMorgan's Law
Thus, $x \notin A \wedge x \notin B$	Definition of $\notin$
Hence, $x \in \complement A\ \wedge x \in \complement B$	Definition of $\complement$
Thus, $x \in \complement A \cap \complement B$	Definition of $\complement A \cap \complement B$
Therefore, $\complement(A \cup B) \subset \complement A \cap \complement B$	

(ii) Let $x \in \complement A \cap \complement B$

Therefore, $x \in \complement A \wedge x \in \complement B$	Definition of $\complement A \cap \complement B$
Thus, $\sim(x \in A\ ) \wedge \sim(x \in B)$	Definition of $\complement$
Hence, $\sim(x \in A \vee x \in B)$	DeMorgan's Law
Therefore, $\sim(x \in A \cup B)$	Definition of $A \cup B$
Thus, $x \in \complement(A \cup B)$	Definition of $\complement$
Hence, $\complement(A \cup B) \subset \complement A \cap \complement B$	
Therefore, $\complement(A \cup B) = \complement A \cap \complement B$	

2. Proof 2 (using the definition of $\complement(A \cup B)$ and $\complement A \cap \complement B$)

$\complement(A \cup B)$	$= \{x \mid x \notin A \cup B\}$	Definition of $\complement$
	$= \{x \mid \sim(x \in A \cup B)\}$	Definition of $\notin$
	$= \{x \mid x \sim(x \in A \vee x \in B)\}$	Definition of $\cup$
	$= \{x \mid x \in \sim(x \in A\ ) \wedge \sim(x \in B)\}$	DeMorgan's Law
	$= \{x \mid x \notin A \wedge x \notin B\}$	Definition of $\notin$
	$= \{x \mid x \in \complement A\ \wedge\ x \in \complement B\}$	Definition of $\complement$
	$= \{x \mid x \in \complement A \cap \complement B\}$	Definition of $\complement A \cap \complement B$
	$= \complement A \cap \complement B$	

Proof of *i:* $A \cap \complement A = \varnothing$

$A \cap \complement A$	$= \{x \mid x \in A \cap \complement A\}$	
	$= \{x \mid x \in A \wedge x \in \complement A\}$	Definition of $A \cap \complement A$
	$= \{x \mid x \in A \wedge x \notin A\}$	Definition of $\complement A$
	$= \varnothing$	Contradiction

Proof of *i*: $A \cup \complement A = X$

$A \cup \complement A$	$= \complement(\complement A \cap A)$	Theorem n above
	$= \complement(\varnothing)$	Theorem o above
	$= X - \varnothing$	Definition of $\complement$
	$= X$	

2.2 **Example**

A high school statistical study of modern language enrollment revealed the following results:

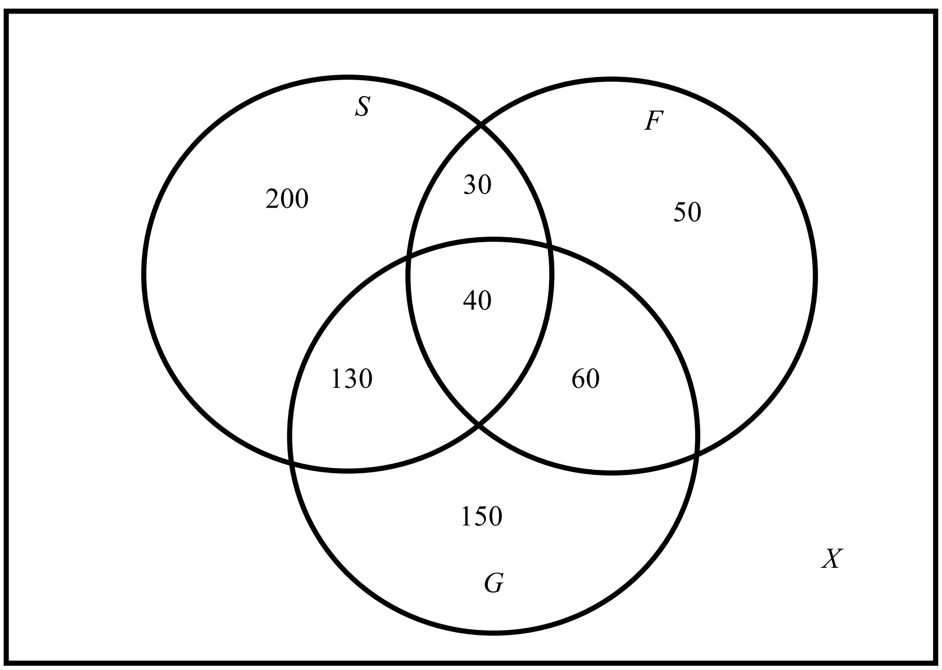
1.	Spanish	400
2.	German and French	100
3.	German only	150
4.	Spanish only	200
5.	Spanish and French, but not German	30
6.	French and German, but not Spanish	60
7.	French only	70

We will answer the following questions using the data above:

i. How many students are enrolled in at least one of the above modern languages?
ii. How many students are enrolled in all three languages?
iii. How many students are enrolled in French?
iv. How many students are enrolled in German?
v. How many students are enrolled in French or German, but not Spanish?

Let X be the body of students surveyed.
Let S be the set of students enrolled in Spanish.
Let G be the set of students enrolled in German.
Let F be the set of students enrolled in French.

The numbers above can then be assigned to sets as follows:



Thus, the above questions can be answered:

i. How many students are enrolled in at least one of the above modern languages?
$200+30+40+130+50+60+150=660$

ii. How many students are enrolled in all three languages?
40

iii. How many students are enrolled in French?
$50+60+40+30=180$

iv. How many students are enrolled in German?
$150+60+40+130=380$

v. How many students are enrolled in French or German, but not Spanish?
$150+60+50=260$

Logic Exercises

1. Write each of the following statements in standard form.

   a. It is not the case that a firetruck is orange.

   ____

   b. It is not the case that a diamond is green.

   ____

   c. It is not the case that a butterfly is not an insect.

   ____

   d. It is not the case that Los Angeles is in Maryland.

   ____

   e. It is not the case that a dog is not a mammal.

   ____

2. Write the negation in standard form of each statement above.

   ~a: ______________________________

   ~b: ______________________________

   ~c: ______________________________

   ~d: ______________________________

   ~e: ______________________________

3. Write the negation in standard form for each of the following statements.

Statement: a: Hemingway was a writer.
Negation: ~a: ______________________________

Statement: b: Benito Juarez was a liberator.
Negation: ~b: ______________________________

Statement: c: Martin Luther King was a civil rights advocate.
Negation: ~c: ______________________________

Statement: d: Snoopy is a cat.
Negation: ~d: ______________________________

Statement: e: A whale is not small.
Negation: ~e: ______________________________

Statement: f: Cinco de Mayo is not in September.
Negation: ~f: ______________________________

Statement: g: Sulfur is a metal.
Negation: ~g: ____

Statement: h: An apple is not blue.
Negation: ~h: ____

4. Give the truth values (True or False) of each statement and its negation in #3.

a ________	b ________	c _______	d _______
~a ________	~b ________	~c _______	~d _______
e ________	f ________	g _______	h _______
~e ________	~f ________	~g _______	~h _______

5. Given the statements

a: Santa Claus is jolly.
b: An oak is a tree.
c: Austin is the capitol of Kansas.

i. Write the conjunction of a and b.
$a \wedge b$: ____.

ii. Write the disjunction of a and c.
$a \vee c$: ____.

iii. Write the conjunction of ~a and b.
$\sim a \wedge b$: ____.

iv. Write the disjunction of ~a and ~b.
$\sim a \vee \sim b$: ____.

v. Write the conjunction of ~a and c.
$\sim a \wedge c$: ____.

vi. Write the disjunction of ~b and ~c.
$\sim b \vee \sim c$: ____.

vii. Write the conjunction of ~a and c.
$\sim a \wedge c$: ____.

viii. Write the disjunction of ~b and ~a.
$\sim b \vee \sim a$: ____.

ix. Write the conjunction of a and ~c.
$a \wedge \sim c$:
____.

6. Write the truth value for each of the statements in #5.

i. _______ ii. _______ iii. ______
iv. ______ v. ________ vi. ______
vii. ______ viii. ______ ix. ______

7. Complete the following truth table:

p	$\sim p$	q	$\sim q$	$p \wedge \sim q$

8. Given the statements.

p: Summer is hot.

q: The pool is cool.

a. Write the conditional statement with p as the antecedent and q as the consequent.

b. Write the converse of the statement in 8a.

c. Write the inverse of the statement in 8a.

d. Write the contrapositive of the statement in 8a.

9. Given the statements.

p: Santa Claus is jolly.

q: An oak is a tree.

r: Austin is the capitol of Kansas.

a. Write in standard form the negation of the conjunction of p and q.

b. Write in standard form the negation of the disjunction of p and q.

c. Write in standard form the negation of the conjunction of p and r.

d. Write in standard form the negation of the disjunction of p and r.

e. Write in standard form the negation of the conjunction of q and r.

f. Write in standard form the negation of the disjunction of q and r.

10. Complete the following truth table.

p	$\sim p$	q	$\sim q$	$\sim p \vee \sim q$	$p \wedge q$	$\sim(p \wedge q)$	$p \vee \sim p$	$p \wedge \sim p$
T	F	T	F					
T	F	F	T					
F	T	T	F					
F	T	F	T					

Is $\sim p \vee \sim q \equiv \sim(p \wedge q)$? Give a brief explanation for your answer.

11. Complete the truth table for $\sim p \wedge \sim q$ and $\sim(p \vee q)$.

p	$\sim p$	q	$\sim q$	$\sim p \wedge \sim q$	$p \vee q$	$\sim(p \vee q)$
T	F	T	F			
T	F	F	T			
F	T	T	F			
F	T	F	T			

Are $\sim p \wedge \sim q$ and $\sim(p \vee q)$ logically equivalent? Give a brief explanation for your answer.

12. Construct a truth table for: $p \wedge \sim q, \sim p \vee q, p \vee \sim q, \sim p \wedge q$

p	$\sim p$	q	$\sim q$	$p \wedge \sim q$	$\sim p \vee q$	$p \vee \sim q$	$\sim p \wedge q$
T	F	T	F				
T	F	F	T				
F	T	T	F				
F	T	F	T				

Which statements are negations of each other?

13. Prove $p \to q$ is not logically equivalent to $q \to p$.

14. Prove $\sim\left(p \to q\right)$ is not logically equivalent to $\sim p \to q$.

15. Complete the truth tables.

p	q	r	$p \wedge q$	$q \wedge r$	$p \vee q$	$q \vee r$	$\left(p \wedge q\right) \vee r$	$p \wedge \left(q \vee r\right)$	$\left(p \vee q\right) \vee r$	$p \vee \left(q \vee r\right)$

Which statements are logically equivalent?

16. Use a truth table to prove: $\left[\left(\sim p \vee q\right) \wedge p\right] \vee q \equiv q$

17. Prove $\left[\left(\sim p \vee q\right) \wedge \sim q\right] \vee \sim p \equiv \sim p$

18. Prove $(p \wedge q) \vee q \equiv q$

19. Prove $(p \vee q) \wedge p \equiv p$

20. Prove $p \vee q \equiv (p \vee q) \vee p$

21. Prove $\sim(p \vee q) \equiv \sim p \wedge \sim q$

22. Prove $\sim(p \wedge q) \equiv \sim p \vee \sim q$

23. Prove $p \rightarrow q \equiv \sim q \rightarrow \sim p$

24. Construct a truth table for $p \rightarrow (q \rightarrow r)$ and $(p \wedge q) \rightarrow r$

Are these statements logically equivalent? Justify your answer.

25. Prove: $p \rightarrow p$ is a tautology

26. Prove $[(p \rightarrow q) \wedge p] \rightarrow q$ is a tautology

27. Prove $\left[\left(p \to q\right) \wedge q\right] \to p$ is not a tautology

28. Prove: For any statement p, $p \Rightarrow p$

29. Prove $\left[\left(p \to q\right) \wedge p\right] \Rightarrow q$

Note: There are two ways to define implication. Either can be used here.

30. Prove $\left[(p \to q) \wedge \sim p\right] \not\Rightarrow \sim q$

31. Test the validity of the following arguments:

a.
$$\begin{array}{l} P_1 : p \to q \\ \underline{P_2 :\ q} \\ C :\ p \end{array}$$

b.
$$\begin{array}{l} P_1 :\ p \to q \\ \underline{P_2 :\ \sim q} \\ C :\ \sim p \end{array}$$

c.
$$\begin{array}{l} P_1 :\ p \vee q \\ \underline{P_2 :\ p \vee \sim q} \\ C :\ p \end{array}$$

In Exercises 32-37:

(a) rewrite each argument in symbolic form using the suggested letters.

(b) check the validity of each argument.

32. The number can be odd (o) or it can be even (e). If it is even, it can be divided evenly by 2 (d). But the number cannot be divided evenly by 2. Therefore, the number is odd.

33. If it rains (r), then the train is late (l). The train is not late. Therefore, it is not raining.

34. A whale is a mammal (m). If a whale is a mammal and a spider has six legs (s), then a frog is a reptile (r). But, a frog is not a reptile. Therefore, a spider does not have six legs.

35. If I cannot see the moon (m), then it is cloudy (c). If it is cloudy, then I cannot see the stars (s). Therefore, if I can see the stars, then I can see the moon.

36. The day is a weekday (w), or there are no classes (c). If there is a movie matinee (m), then it is not a weekday. Therefore, if there are classes, there is no movie matinee.

37. If the juniper is a tree (t), it will lose its leaves in the winter (w). The juniper does not lose its leaves in the winter. Therefore, the juniper is not a tree.

In Exercises 38-40:

(a) define letters to represent each statement.
(b) rewrite each argument in symbolic form using your selected letters.
(c) check the validity of each argument.

38. If the chemical is an acid, it will have a pH lower than 7. The chemical is an acid or a base. the pH is not lower than 7. Therefore, the chemical is a base.

39. A fever can be caused by a virus or by a bacterial infection. If it is caused by a bacterial infection, it needs to be treated with antibiotics. The fever does not need to be treated with antibiotics. Therefore, the fever is caused by a virus.

40. If the filament is made of copper, it will conduct electricity. The filament is made of copper or it is made of nylon. If it is made of nylon, it will melt when it is heated. Therefore, if the filament does not conduct electricity, it will melt when it is heated.

41. Prove that the following argument is valid.

$$\begin{array}{l} P_1:\ (p \vee q) \to r \\ P_2:\ r \to (s \vee t) \\ P_3:\ t \to w \\ \underline{P_4:\ \sim s \wedge \sim w} \\ C:\ \sim p \wedge \sim q \end{array}$$

For questions 42-46:

Let D be the predicate "*is a Democrat*. "
Let R be the predicate "*is a Republican*. "
Let C be the predicate "*is conservative*. "
Let S be the predicate "*is a statesman*. "
Let P be the predicate "*is a politician*. "

42. Given the statement: *All Republicans are conservative.*

a. Write it in symbolic form.

b. Write its negation symbolically.

c. Translate the negation.

43. Given the statement: *Some politicians are statesmen*.

a. Write it in symbolic form.

b. Write its negation symbolically.

c. Translate the negation.

44. Given the statement: *Some Democrats are not politicians*.

a. Write it in symbolic form.

b. Write its negation symbolically.

c. Translate the negation.

45. Given the statement: *All politicians are not Republicans*.

a. Write it in symbolic form.

b. Write its negation symbolically.

c. Translate the negation.

46. Write the negation of the following statements.

(a) All Republicans are conservative and some politicians are statesmen.

(b) Some Democrats are not politicians or all Republicans are conservative.

(c) If some Democrats are not politicians, then all Republicans are conservative.

(d) If all politicians are not Republicans, then all Republicans are conservative.

47. Write the symbolic form for the following argument.

All jewels are precious.
A diamond is a jewel.
Therefore, a diamond is precious.

48. Construct a proof for the argument in Exercise 47.

49. Write the symbolic form of the argument.

All whales are mammals.
An oyster is not a mammal.
Therefore, an oyster is not a whale.

50. Construct a proof for the argument in Exercise 49.

For exercises 51 and 52:

Let I be the predicate "*is a pie.* "
Let Q be the predicate "*is a square.* "
Let T be the predicate "*is a triangle.* "
Let G be the predicate "*tastes good.* "

51. Given the statements

p: Some pies are square.
q: All squares are not triangular.
r: Some pies are not triangular.
s: All pies taste good.

(a) Write each in symbolic form.

(b) Write the negation of each symbolically.

(c) Translate the negations.

52. Negate the following statements.

(a) `All pies taste good and some pies are square.

(b) Some pies are square or all squares are not triangular.

(c) If some pies are square, then all pies taste good.

ELEMENTARY SET THEORY EXERCISES

1. Let

$$X = \{1,2,3,4,5,6,7,8,9,10\}$$
$$A = \{1,4,7\}$$
$$B = \{1,2,3,6,7\}$$
$$C = \{3,6,7,8,9\}$$

Find:

(a) $A \cap B$

(b) $A \cup C$

(c) $A - C$

(d) $A \cup (B \cap C)$

(e) $A \cap (B \cap C)$

(f) $A \cap (B \cup C)$

(g) $B - C$

(h) $(B - C)'$

(i) $A' \cup B'$

(j) $A' \cap B'$

(k) $(A \cup B) \cap (A \cup C)$

(l) $A' \cap B$

2. Let $P(X)$ be the set of all subsets of X. $P(X)$ is called the power set of X.

For example, if $X=\{1,2\}$, then $P(X)=\{\varnothing,\{1\},\{2\},\{1,2\}\}$.

(a) Let $X=\{1,2,3\}$. List the elements of $P(X)$.

(b) Let $X=\{1,2,3,4\}$. List the elements of $P(X)$.

(c) Let $X=\{1,2,3,4,5,6,7,8,9,10\}$

(i) Let $W(X)=\{A\subset X \mid \text{the sum of distinct elements of } A \text{ is } 15\}$

List the elements of $W(X)$. (Note that $W(X)\subset P(X)$)

(ii) Let $K(X)=\{A\subset X \mid \text{ the sum of the distinct elements of } A \text{ is } 50\}$

List the elements of $K(X)$.

(iii) Let $G(X)=\{A\subset X \mid \text{ the sum of the distinct elements of } A \text{ is } 55\}$

List the elements of $G(X)$.

In Exercises 3 throught 13, let *A, B*, and *C* be arbitrary subsets of a fixed set *X*.

3. Prove: $A \cap (B \cap C) = (A \cap B) \cap C$

4. Prove: $A \cup (B \cup C) = (A \cup B) \cup C$

5. Prove: $A \cup B = B \cup A$

6. Prove: $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

7. Prove: $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

8. Prove: If $A \subset B$, then $A \cup B = B$

9. Prove: If $A \cup B = B$, then $A \subset B$

10. Prove: If $A \subset B$, then $A \cap B = A$

11. Prove: If $A \cap B = A$, then $A \subset B$

12. Prove: $\complement(A \cap B) = \complement A \cup \complement B$

13. Prove: $A \subset A \cup B$

14. In a survey of 300 San Antonians, 130 claimed they read the news in the newspaper and 150 claimed they watched the news on TV. Sixty said they read both the internet news reports and the newspaper while 50 read only the internet news, 30 read only the newspaper, 60 only watched the news on TV, and 20 did all three.

(a) How many read both the internet news reports and the newspaper but did not watch the news on TV?

(b) How many read the newspaper <u>and</u> watched the news on TV, but did not read the internet news reports?

(c) How many read the internet news <u>and</u> watched the news on TV?

(d) How many did not read or watch the news at all?

(e) How many read the internet news reports?

15. Five hundred San Antonians were asked to reveal their restaurant preferences:

190 liked McDonald's
170 liked Luby's
50 liked both Luby's and McDonald's
90 liked both Luby's and Mi Tierra
100 liked only Mi Tierra
70 liked only McDonald's
60 liked both Luby's and Mi Tierra but not McDonald's

(a) How many like all three restaurants?

(b) How many liked only Luby's?

(c) How many liked both Mi Tierra and McDonald's, but not Luby's?

(d) How many liked Mi Tierra?

(e) How many liked none of the restaurants?

16. A magazine vendor noted the following sales statistics in a given week:

150 customers purchased Time
200 customers purchased Newsweek
50 customers purchased all three
400 customers purchased none of these magazines
30 customers purchased Time but none of the others
40 customers purchased Consumer Reports but none of the others
60 purchased both Consumer Reports and Newsweek but not Time
20 purchased both Consumer Reports and Time but not Newsweek

(a) How many customers purchased only Newsweek?

(b) How many customers purchased both Newsweek and Consumer Reports but not Time?

(c) How many purchased either Time or Newsweek but not both?

(d) How many customers did the vendor have during the week?

17. Persons are classified according to blood type and Rh quality by testing a blood sample for the presence of three antigens: A, B, and Rh. Blood is of type AB if it contains antigens A and B; A if it contains A, but not B; B if it contains B, but not A; and O if it contains neither A nor B. In addition, blood is classified as Rh positive (+) if the Rh antigen is present and Rh negative (-) otherwise. A national report stated that if 1000 people were picked at random and their blood were sampled, then statistics indicated that 490 would contain antigen A, 515 would contain antigen B, 430 would contain Rh, 205 would contain A and B, 140 would contain A and Rh, 160 would contain B and Rh, 60 would contain all three. On the basis of this information, determine which of the eight blood classifications is the most plentiful and which is the rarest.

18. The results of a questionnaire sent to all the alumni of a certain university showed that 66 percent expected to attend the forthcoming activities celebrating the 150^{th} anniversary of the school's founding. On the day of the festivities, it was learned that 90 percent of those who actually *did* attend had previously indicated that they *would* attend, while 30 percent fo those who did *not* attend had previously indicated that they *would* attend. Determine (a) what percent of the total alumni attended and (b) what percent of the total alumni acted differently than they had first indicated.

Hint: For simplicity, assume the total number of alumni is 100.